Department of Mechanical and Aerospace Engineering

MAE 334 – Introduction to Computers and Instrumentation

Laboratory 5 - Linear Systems, Fourier Transforms and the Transfer Function

Objective:

To measure and calculate the transfer function of the simple single pole passive RC filter using several different methods. Basic Fourier analysis techniques will be introduced.

Background:

Most passive circuits and many mechanical systems encountered in the laboratory can be treated as linear systems. In general, the output of a linear system can be written as some weighted history of the input. If $h_1(\Delta)$ is used to denote the system's "memory" function we can write:

$$y(t) = \int_{-\infty}^{t} h_1(t-\Delta)F(\Delta)d\Delta \tag{1}$$

where $F(\Delta)$ is the input signal and $y(t)$ is the output. This is known as a convolution integral in time. Usually the "memory" of the system fades with time which means that $h_1(t-\Delta)$ approaches 0 as Δ goes to infinity. Since our linear systems cannot foretell or anticipate the future, $h_1(t)$ is always zero for times beyond the present time. For convenience we can define a new function $h(\Delta)$ over the entire range ($-\infty < \Delta < +\infty$) so that

$$h(\Delta) = \begin{cases} h_1(\Delta), & \Delta \geq 0 \\ 0, & \Delta < 0 \end{cases} \tag{2}$$

We can now use this in equation (1) to extend the integral for all time

$$y(t) = \int_{-\infty}^{+\infty} h(t-\Delta)F(\Delta)d\Delta \tag{3}$$

A First Order System (Low Pass Filter)

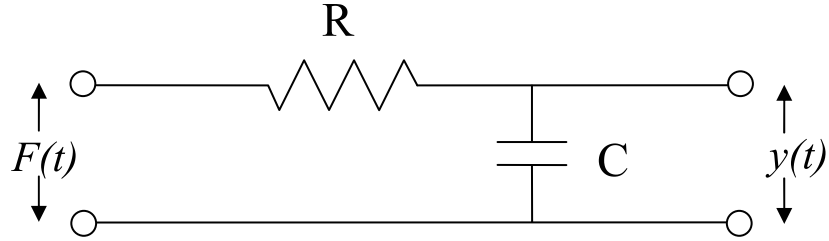


Figure 1. A single pole passive low pass filter, where *F(t)* is the input signal and *y(t)* is the output of the filter. Often referred to as an RC or single pole Butterworth filter.

The low pass filter drawn in Figure 1 can be modeled as a first order differential equation of the form:

$$RC\frac{dy}{dt} + y = F$$

If *F(t)* is the input signal and *y(t)* the output. This equation has the following general solution:

$$y(t) = \int_{-\infty}^{t} F(\Delta)\left[\frac{1}{RC}e^{-(t-\Delta)/RC}\right]d\Delta \qquad (4)$$

following equations (1), (2) and (3) we conclude

$$h(t) = \begin{cases} \frac{1}{RC}e^{-t/RC} & \ldots \quad t \geq 0 \\ 0 & \ldots \quad t < 0 \end{cases} \qquad (5)$$

You should recognize this as a standard first order system response function of the same form as used to describe the thermocouple in lab 2 and the calorimeter in lab 3. In this case the time constant, τ, would be *RC*, the resistance and capacitance values.

The Impulse Response of the Low Pass Filter

Recall that the <u>delta</u> or <u>impulse function</u>, *δ(t)*, is zero at all times except *t* = 0, where it is infinite and that it is infinite in such a way that integral for any time that includes *t=0* is 1.

$$\int_{-\varepsilon}^{+\varepsilon} \delta(t)dt = 1$$

Therefore ε can be as small as we please. If we subject our filter to an impulse function, *F(t)= δ(t),* we will observe the <u>impulse response function</u> at the output, *y(t).* Using our impulse function as the input signal in equation (3) and the properties of *δ(t)* we have for the output

$$y(t) = \int_{-\infty}^{+\infty} h(t-\Delta)F(\Delta)d\Delta = \int_{-\infty}^{+\infty} h(t-\Delta)\delta(\Delta)d\Delta = h(t)$$

Remember the integral is zero everywhere except at Δ = 0. The output from the system subjected to the impulse, *h(t),* is usually called the <u>impulse response function</u>.

If *F(t)= δ(t)* is substituted into the general solution (4), (remembering that the integration of the impulse function merely selects the value of the integrand at Δ =0) we obtain the impulse response as:

$$y(t) = \begin{cases} \frac{1}{RC}e^{-t/RC} & \ldots \quad t \geq 0 \\ 0 & \ldots \quad t < 0 \end{cases}$$

This solution is just like *h(t)* obtained above.

Because this low pass filter can be described by a well behaved continuous function and the input to the filter is a continuous function the Fourier transform can be found. ***The Fourier transform of a convolution is simply the product of the Fourier transform of the two functions involved in the convolution***, that is:

If

$$H(f) = \text{Fourier Transform of } \{h(t)\}$$
$$\hat{y}(f) = \text{Fourier Transform of } \{y(t)\}$$
$$\hat{F}(f) = \text{Fourier Transform of } \{F(t)\}$$

then the frequency response function of the filter can be expressed as

$$\hat{y}(f) = H(f)\hat{F}(f) \tag{6}$$

The linear system simply takes each Fourier component (frequency) of the input signal and multiplies it by a gain which is frequency dependent. Note that since *H(f)* is generally complex, the phase of each Fourier component can also be changed. (This in effect scrambles the signal in time by delaying some frequency parts of the signal differently with respect to other parts. If you were to send your voice signal into such a filter it would sound lower and slightly distorted.)

Obviously equation (6) represents a great simplification over equation (3) since it involves only multiplication instead of integration. Therefore *H(f)* is a great thing to know since it tells us everything we need about how the system affects input signals. Engineers often refer to the frequency response function as the transfer function.

Frequency Response Function of a Low Pass Filter

From the definition of the Fourier transform,

$$H(f) = \text{Fourier Transform of } \{h(t)\}$$
$$H(f) = \int_{-\infty}^{+\infty} h(t)e^{-i2\pi ft}dt = \int_{0}^{+\infty} \frac{1}{RC}e^{-t/RC}e^{-i2\pi ft}dt = \left[\frac{1}{1+i2\pi fRC}\right]$$

We can break this into the amplitude and phase response

$$\left|H(f)\right| = \frac{1}{\left[(2\pi fRC)^2+1\right]^{1/2}} = \frac{\left|y(f)\right|}{\left|F(f)\right|} \tag{7}$$

$$\tan(\phi) = -2\pi fRC \tag{8}$$

If we define the filter cutoff frequency as $f_c = 1/(2\pi RC) = 1/(2\pi\tau)$ and substitute this into equations (7) and (8) at we get

$$\left|H(f_C)\right| = \frac{1}{\sqrt{2}} = 0.707 = -3\text{ dB} \tag{9}$$

and

$$\phi(f_c) = \tan^{-1}(1) = 45° \tag{10}$$

At the filter cutoff frequency the amplitude of the input signal is reduced to 0.707 of the input and the phase is shifted by 45 degrees. The transfer function is commonly displayed on a Bode plot in decibels (dB) = 20 $\log_{10}|H(f)|$ vs. $\log_{10}(f/f_c)$, see Figure 2. Pay particular attention to the -3 dB point on the amplitude plot and the -45 degree point on the phase angle plot.

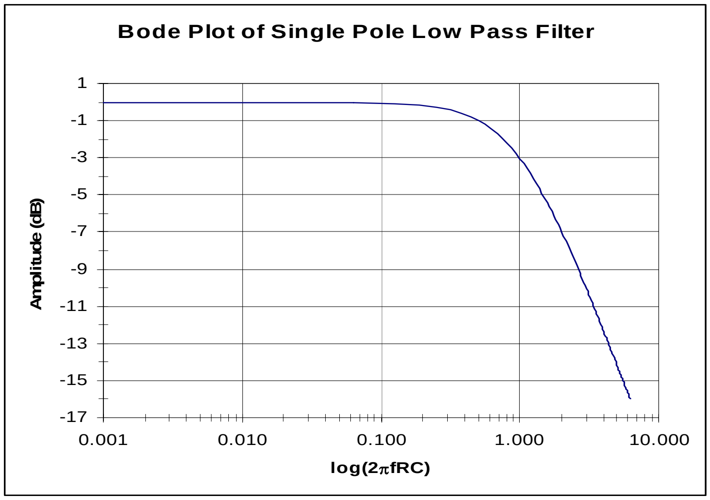


Figure 2. Amplitude of the frequency response function of a passive single pole low pass filter.

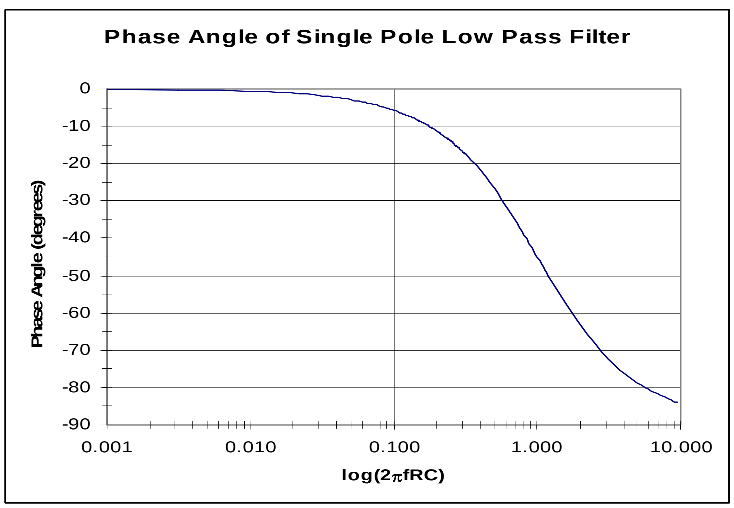


Figure 3. Phase angle of the frequency response function of a passive single pole low pass filter.

Unfortunately it is not always convenient to create an impulse in the laboratory. Besides, it is often the frequency response function which is desired anyway. Therefore it is useful to inquire as to whether the frequency response function can be obtained directly. It is important to remember that both the frequency response function and the impulse response function contain the same information – if either is known, both are known since they are a Fourier transform pair. Using the properties of the delta function and Fourier transform it can be shown that the output to a sine wave input is:

$$y(t) = |H(f_0)| \cos\left[2\pi f_0 t + \phi(f_0)\right] \quad (11)$$

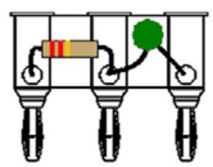
Hence the output is simply the input delayed by a time $\phi(f_0)/2\pi f_0$ and multiplied by $|H(f_0)|$. Therefore by taking the ratio of output to input peak voltages we can easily determine $|H(f_0)|$. By using different frequencies we can get $|H(f)|$ for all of these frequencies. By measuring the time delay between the two signals (input and output) and setting the delay equal to $\phi(f_0)/2\pi f_0$, we can find the phase angle function $\phi(f)$ for all of these frequencies as well. Do to the limitations of our ADC hardware it is very difficult to determine the phase lag accurately so we will not record this information. Given a high speed simultaneous sampling ADC or oscilloscope, this time delay can be easily measured.

Procedure

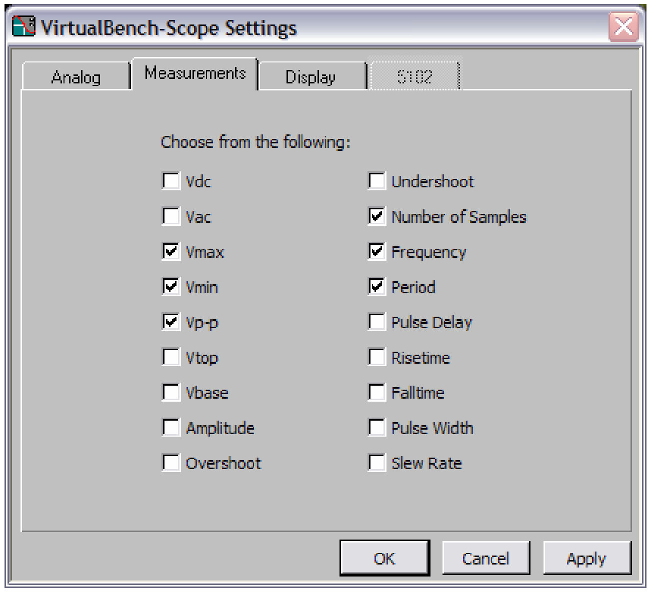
In this lab we will obtain the frequency response function of a passive single pole filter in a couple of different methods. The first method will involve inputting a number of different frequency sine waves, $F(f_0)$, into the filter and measuring the output, $y(t)$. By using equation (11) above the magnitude of the transfer function, $|H(f_0)|$, can be found. The second method will involve recording the response of the filter to a step input and obtaining the time constant, τ , of the filter. Then using this time constant perform an Excel simulation of the type of filter you tested to obtain, $|H(f)|$ and $\phi(f)$.

Part 1

1. Sketch the wiring connection for your lab setup. Pay particular attention to the circuit path through the black 3 prong banana connector with the RC circuit on it. Are you testing a low pass or high pass filter? The green component is a capacitor, the cylindrical component is a resistor. Which channel is the filter input, which is the filter output!?



2. Invoke the Virtual Bench Oscilloscope as in previous labs.
3. Enable channel 2, the input to your filter.
4. Enable measurement of channel 2. In the Measurements tab of the Edit, General Settings pull down make sure the check boxes for Vmax, Vmin, Vp-p, Number of Samples, Frequency and Period are enabled.



5. While observing the measurement results for channel 2 on the Oscilloscope tune the function generator to a sine wave frequency of 10.x Hertz and an amplitude of 1.0x volts peak to peak (Vp-p) This is the input signal, *F(f)*, to your filter at *f*=10 Hz.

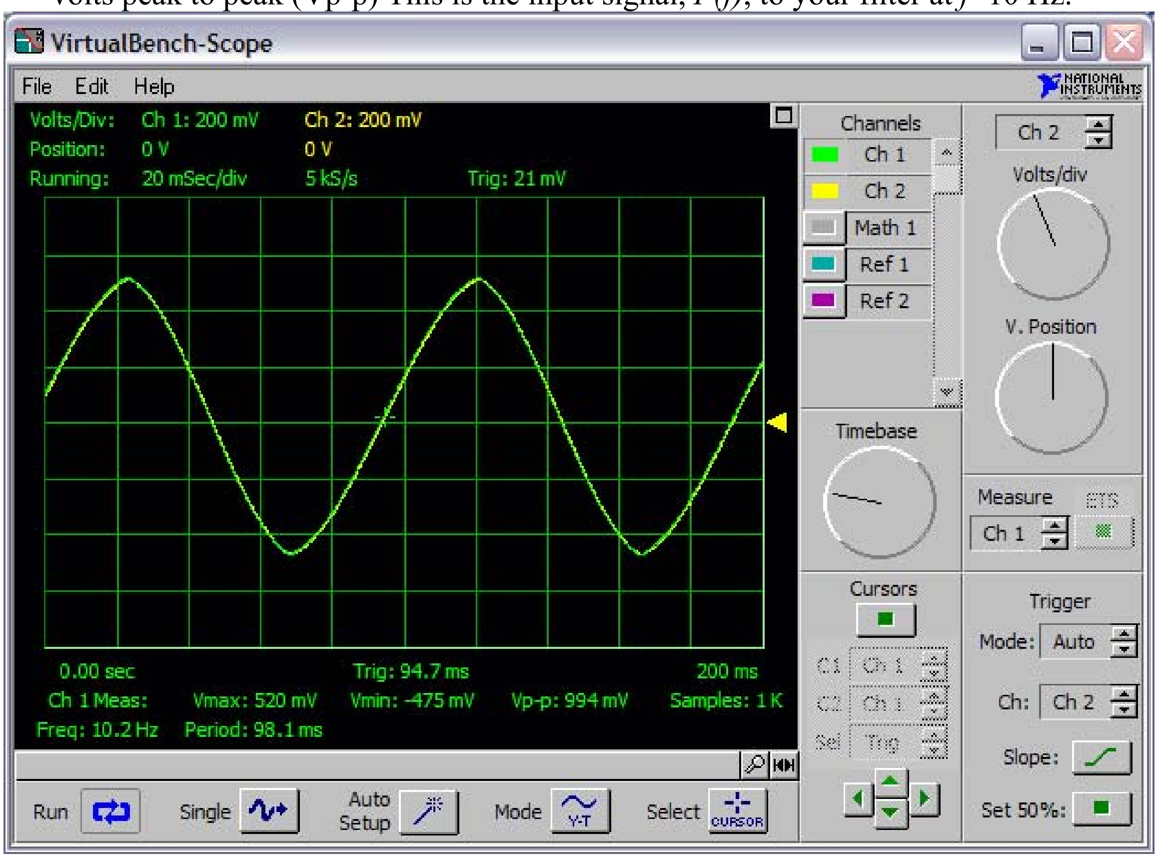


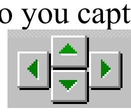
Figure 4 Oscilloscope setup for sine wave input response sweep.

6. Enable channels 1 and 2. Toggle the Measure between channel 1 and 2. Tabulate the values of the Vp-p for both channels which will be used to ultimately obtain the magnitude ratio of the output over the input, $V_{p\text{-}p}(out)/V_{p\text{-}p}(in)$ versus frequency, *f*.

Record these 2 value for a series of approximately 50 frequencies. You will find that the input $V_{p\text{-}p}$ will change slightly as you change the frequency. You do not need to readjust it, but you do need to record it!

Before you begin vary the frequency over a wide range to determine at which point the output magnitude ratio is at the -3 dB point ($V_{p\text{-}p\_out}/ V_{p\text{-}p\_in}$=0.707). This is approximately equal to your filter cutoff frequency, f_c. ***Log this value in your notebook.*** In order to obtain a reasonably smooth transfer function curve concentrate on frequencies close to f_c (take many closely spaced frequency points near f_c and less closely spaced frequency points away from f_c). Be sure to take a frequency point at slightly above 1.x Hz and another as high as your ADC system will allow and still produce a smooth sine wave reconstruction. (5-15 KHz) When you sample the high frequency signals you will get better results if you enable only 1 channel at a time. (This allows the ADC to sample at its maximum rate of for 1 channel instead of half that rate for 2 channels.)

Part 2

1. Change the input waveform on the function generator to a square wave.
2. Set the trigger to channel 1, Norm. Enable only channel 1, the filter output.
3. Watch the output response of the filter to the square wave input. Make sure the square wave frequency is slow enough so you only see the rise of the square wave. The magnitude of the input step function can be increased to better utilize the range of the ADC thus minimizing quantization errors. Remember the ADC has a maximum range of +/- 5 V. Your plot should like the one shown below.
4. Change the oscilloscope buffer size to 1k samples.
5. Set the oscilloscope sampling frequency to as fast as it will go and adjust the voltage range so you capture the response of the filter to the step input. You can use the arrow buttons  below in the cursor control box to move the trigger point close to the start of the oscilloscope trace by clicking on the left arrow. The up and down arrows will adjust the trigger level, which in this screen shot is at 11 mV. You may find that first pressing the Auto Setup button 

will make the adjustment quicker.

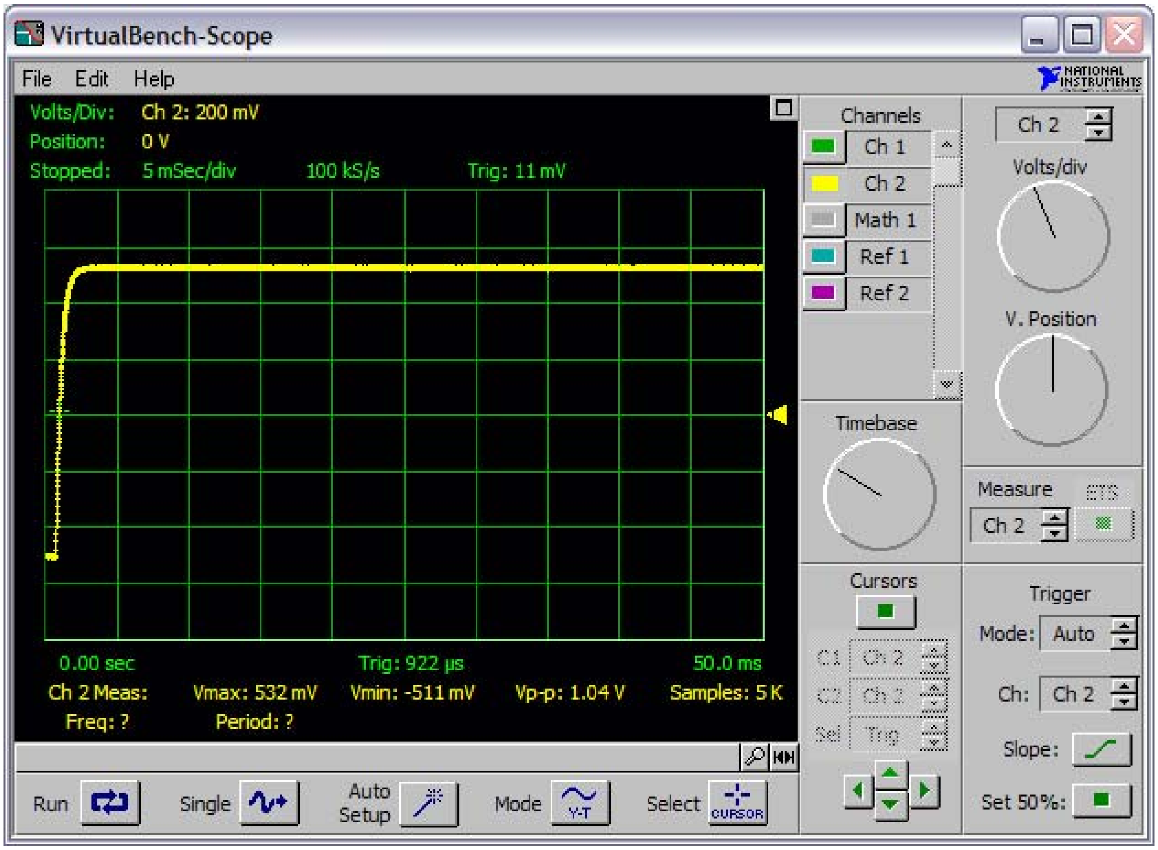


Figure 5 Low Pass Filter Step Input Response Function

6. Enable both channels 1 and 2. Record a “Single” trace and save the input and output step functions. (File, Save Waveforms…)
7. Change the frequency of the step input function so you can see at least 1 full period of the square wave on the scope. Sketch the waveforms in your lab notebook.

RESULTS and DISCUSSION

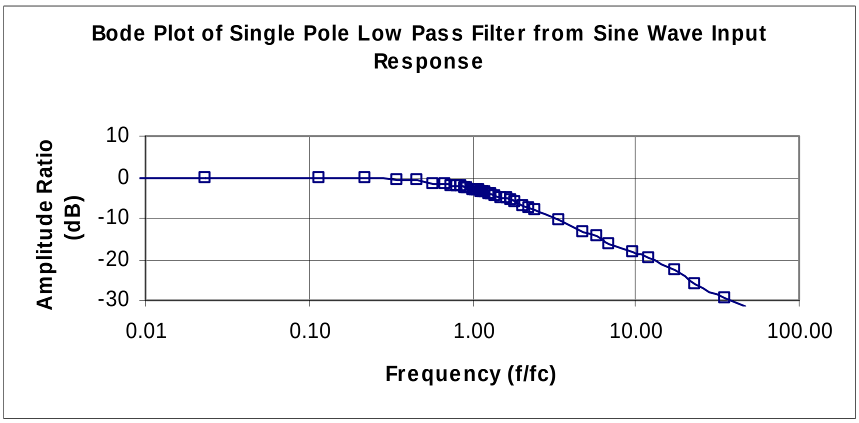
There is an excellent Practical Labcourse and Web Experiments [Java], http://www.st-andrews.ac.uk/~www_pa/Scots_Guide/experiment/intro.html, Course written by and copyright, © 1995-2002, by J. C. G. Lesurf (jcgl@st-and.ac.uk). University of St Andrews, Scotland which simulates with Java the sine wave input response portion of the lab. It is strongly advised that you visit this site and play with the high and low pass filter response Java simulation to obtain an intuitive feel for the response of these types of filters.

Results: Using Excel add to the table from Part 1 above the magnitude ratio in dB and a normalized frequency column, f/f_c.

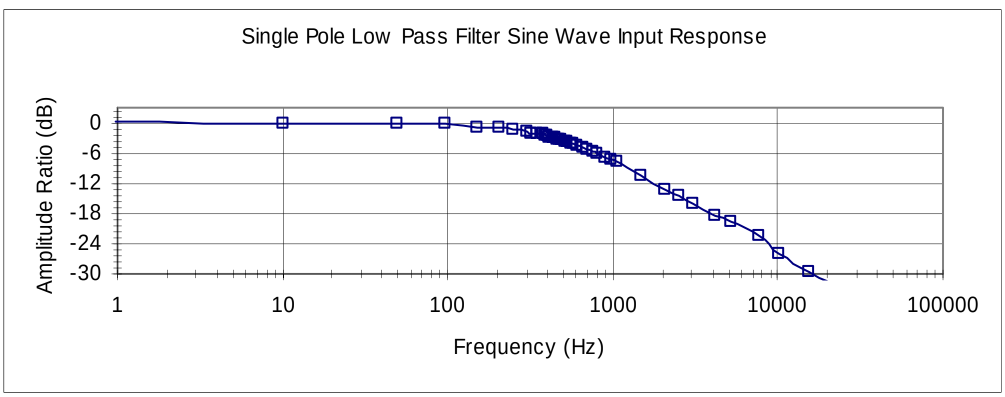
1. Include the table in your lab report.

Frequency (Hz)	Vp-p In	Vp-p Out	Vout/Vin	f/fc	Ratio (dB)
0.962	0.990	0.993	1.003	0.002	0.026
10.200	0.991	0.990	0.999	0.023	-0.009
50.000	1.030	1.010	0.981	0.114	-0.170

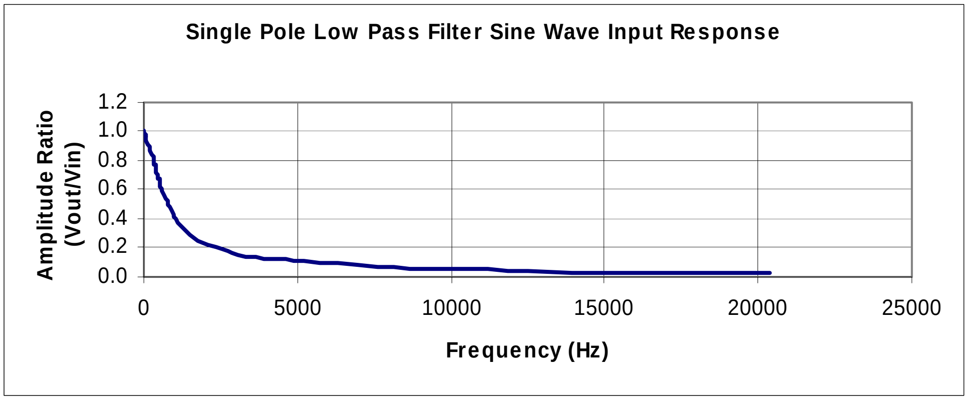
2. Create an amplitude Bode plot (Amplitude in dB vs. log(f/f_c))



3. Create an amplitude ratio in dB vs. log(f) in Hertz plot



4. Create an amplitude ratio V_{out}/V_{in} vs. frequency in Hertz plot



5. Plot the raw data voltage vs. time the filter input and output signals. These should be on the same plot. Zoom in on the time near the step input to better visualize the differences.

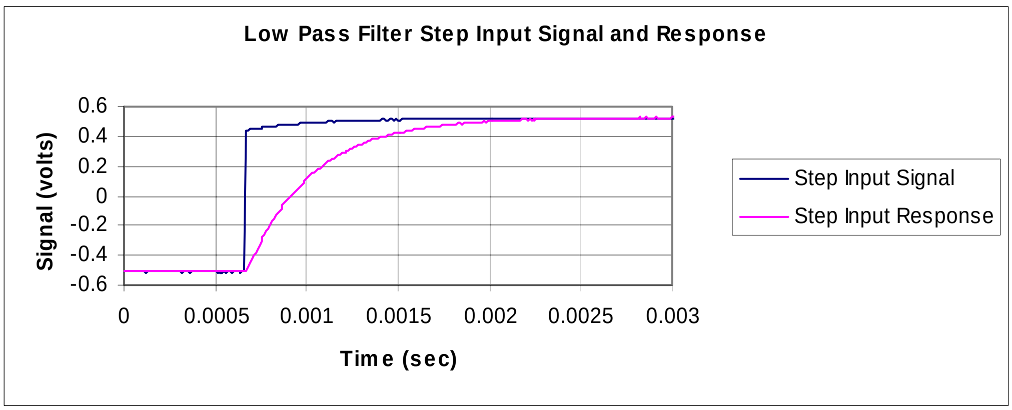
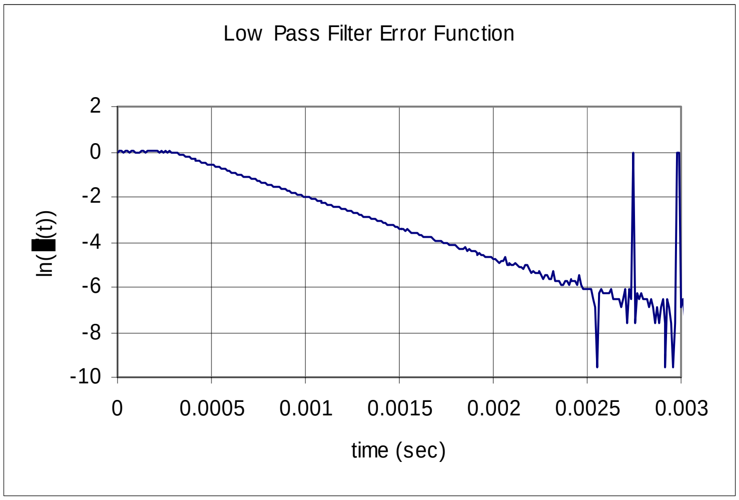


Figure Captions: What is the purpose of a Bode plot? What does it show? Why would you want the *x* axis normalized by the filter cutoff frequency? Contrast and compare the 3 different plots. Note the shape of the step input signal. Was it a true step input? What affect might the input signal spectra have on the method we used to determine the transfer function?

Results (cont.): Using the same technique used in labs 2 and 3 linearize the filter response to the step input function to find the value of the time constant, τ=RC. Using this value of RC create an Excel filter simulation of the filter you tested in the lab. Design your filter simulation so R and C are input variables. See the example spread sheet given on the class web site. ***You must demonstrate and explain this spreadsheet to the TA during your grading session.*** Show up a bit early and login to a computer in the lab and open the spreadsheet.

1. Plot the natural log of the error function, $\ln(\Gamma(t))=\ln[(y(t)-y_f)/(y_0-y_f)]$ vs. time. Find the time constant from the slope of the error function after the start of the step input response.



2. Plot the amplitude ratio in dB vs. log(*f*) of your simulated filter along with the amplitude ratio in dB from Part 1 vs. log(*f*).

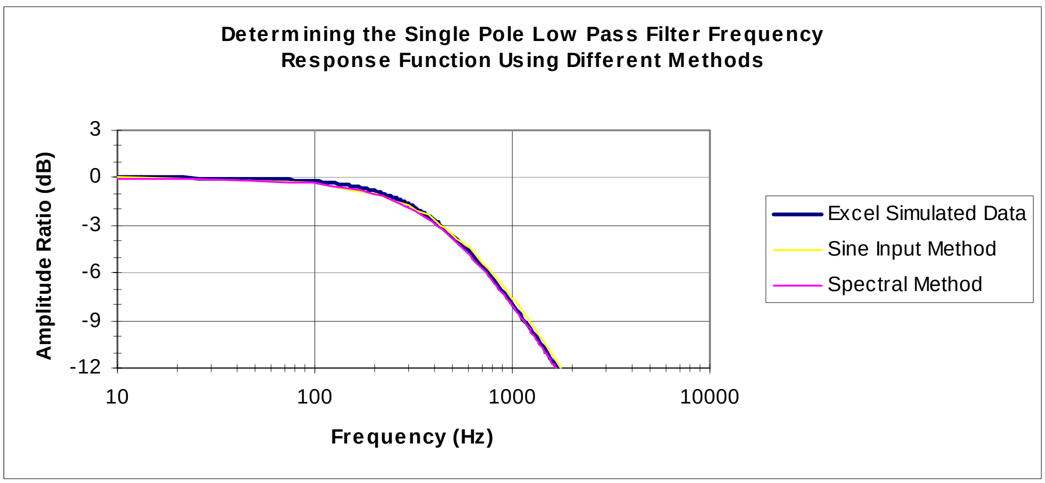


Figure Captions: What was the quality of your estimation of the time constant? Compare and contrast the amplitude vs. frequency curves. What errors are different in the methods that could explain the differences in the curves? Which method was the easiest? Which method would work best for a complicated transfer function?

Extra Credit (Up to 25 points): Using the Excel Fourier Analysis tool pack add-in or MatLab perform a Fourier Transform on your step input and step input response functions. Start the transform *very close* to the start of the step input. Add extra points to the end of your data set by repeating the last response value and calculating the time column so you can perform a 1024 FFT.

1. Plot the magnitude in dB of the FFT vs. frequency in Hertz. Normalize the magnitude by the first non-zero frequency point in the FFT. (The very first point in the FFT output is the DC, zero frequency, point) See the example spreadsheet. This will not produce the true transfer function because the magnitude ratio should be calculated using the transform of the input signal.

2. Add the Amplitude ratio obtained using the Excel simulation and sine wave input method to the plot.
3. Calculate the Fourier transform of the step input wave. Plot the magnitude in dB of the transform vs. frequency.

Discussion: Compare and contrast the plots? Why are they different? What should the plot of a perfect step input function look like? Why does your plot look so different?